The Linear Algebra of the Pascal Matrix

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ABSTRACT

Pascal's triangle can be represented as a square matrix in two basically different ways: as a lower triangular matrix P_n or as a full, symmetric matrix Q_n. It has been found that the $P_n P_n^T$ is the Cholesky factorization of Q_n. P_n can be factorized by special summation matrices. It can be shown that the inverses of these matrices are the operators which perform the Gaussian elimination steps for calculating Cholesky's factorization. By applying linear algebra we produce combinatorial identities and an existence theorem for diophantine equation systems. Finally, an explicit formula for the sum of the kth powers is given.

DEFINITION. The $(n+1)\times(n+1)$ *Pascal matrix* [1] P_n is defined by

$$P_n(i,j) := \binom{i}{j}, \quad i,j = 0,\dots,n, \qquad \text{with} \quad \binom{i}{j} := 0 \quad \text{if } j > i.$$

Further we define the $(n+1)\times(n+1)$ matrices I_n, S_n, and D_n by

$$I_n := \operatorname{diag}(1,1,\dots,1),$$

$$S_n(i,j) := \begin{cases} 1 & \text{if } j \leqslant i, \\ 0 & \text{if } j > i, \end{cases}$$

$$D_n(i,i) := 1 \quad \text{for } i = 0,\dots,n,$$

$$D_n(i+1,i) := -1 \quad \text{for } i = 0,\dots,n-1,$$

$$D_n(i,j) := 0 \quad \text{if } j > i \text{ or } j < i-1.$$

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The Pascal matrix P_n is characterized by its construction rule:

$$P_n(i,i) := P_n(i,0) := 1 \quad \text{for } i = 0, \ldots, n, \qquad P_n(i,j) := \quad \text{if } j > i,$$

$$P_n(i,j) := P_n(i-1,j) + P_n(i-1,j-1) \qquad \text{for} \quad i,j = 1, \ldots, n.$$

It is easy to see that

$$S_n = D_n^{-1}.$$

EXAMPLE.

$$S_2 D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = I_2.$$

Furthermore we need the matrices

$$\overline{P}_k := \begin{bmatrix} 1 & 0^T \\ 0 & P_k \end{bmatrix} \in \mathbb{R}^{(k+2)\times(k+2)}, \qquad k \geqslant 0,$$

$$G_k := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad k = 1, \ldots, n-1, \quad \text{and} \quad G_n := S_n$$

LEMMA 1.

$$S_k \overline{P}_{k-1} = P_k \qquad \textit{for} \quad k \geqslant 1.$$

Proof. For $k = 1$ we have $\overline{P}_{k-1} = I_k$ and $S_k = P_k$. Let $k > 1$. With the definition of the matrix product and a familiar combinatorial identity we find (see [3, p. 7]) for $j \geqslant 1$

$$S_k \overline{P}_{k-1}(i,j) = \sum_{l=1}^{i} \binom{l-1}{j-1} = \sum_{l=j-1}^{i-1} \binom{l}{j-1} = \binom{i}{j} = P_k(i,j),$$

and for $j = 0$ it follows that $S_k \overline{P}_{k-1}(i,j) = 1 = P_k(i,j)$. ■

EXAMPLE.

$$S_3 \overline{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

An immediate consequence of Lemma 1 and the definition of the G_k's is

THEOREM 1. *The Pascal matrix P_n can be factorized by the summation matrices G_k:*

$$P_n = G_n G_{n-1} \cdots G_1. \tag{1}$$

EXAMPLE.

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

For the inverse of the Pascal matrix we get

$$P_n^{-1} = G_1^{-1} G_2^{-1} \cdots G_n^{-1} = F_1 F_2 \cdots F_n \tag{2}$$

with

$$F_k := G_k^{-1} = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & D_k \end{bmatrix}, \qquad k = 1, \ldots, n-1,$$

and

$$F_n = G_n^{-1} = D_n.$$

Let P_n^* be defined by $P_n^*(i,j) := (-1)^{i+j} P_n(i,j)$ and

$$\bar{P}_k^* := \begin{bmatrix} 1 & 0^T \\ 0 & P_k^* \end{bmatrix} \in \mathbb{R}^{(k+2)\times(k+2)}, \qquad k \geqslant 0.$$

LEMMA 2.

$$\bar{P}_{k-1}^* D_k = P_k^* \qquad \textit{for} \quad k \geqslant 1.$$

Proof. For $k = 1$ we have $\bar{P}_{k-1}^* = I_k$ and $D_k = P_k^*$. Let $k > 1$. By the Pascal matrix construction rule we get for $i \geqslant 1$ and $j \geqslant 1$

$$\begin{aligned} &\left(\bar{P}_{k-1}^* D_k\right)(i,j) \\ &\quad = (-1)^{i-1+j-1} P_{k-1}(i-1, j-1) - (-1)^{i-1+j} P_{k-1}(i-1, j) \\ &\quad = (-1)^{i+j} [P_{k-1}(i-1, j-1) + P_{k-1}(i-1, j)] \\ &\quad = (-1)^{i+j} P_{k-1}(i,j) = (-1)^{i+j} P_k(i,j) = P_k^*(i,j). \end{aligned}$$

For $j = 0$ we have $(\bar{P}^*_{k-1} D_k)(i, j) = (-1)^i = P^*_k(i, j)$, and for $i = 0$, $j \geqslant 1$ we have $(\bar{P}^*_{k-1} D_k)(i, j) = 0 = P^*_k(i, j)$. ∎

EXAMPLE.

$$\bar{P}^*_2 D_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Lemma 2 makes it possible to factorize P^*_n by the difference matrices F_k, and with (2), it follows that

THEOREM 2. *One has*

$$P_n^{-1} = F_1 F_2 \cdots F_n = P^*_n; \tag{3}$$

in particular,

$$P_n^{-1} = J_n P_n J_n, \tag{4}$$

where

$$J_n := \operatorname{diag}\left(1, -1, \dots, (-1)^n\right) \in \mathbb{R}^{(n+1)\times(n+1)}.$$

EXAMPLE.

$$P_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Equation (4) represents the well-known inverse relation [3]

$$\delta_{nk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j} \binom{j}{k}.$$

We define the symmetric Pascal matrix Q_n as

$$Q_n(i,j) := \binom{i+j}{j}, \qquad i,j = 0,\dots,n.$$

Similarly to the Pascal matrix P_n, the elements of Q_n obey the following construction rule:

$$\begin{aligned} Q_n(0,j) &= Q_n(j,0) = 1, \qquad j = 0,\dots,n, \\ Q_n(i,j) &= Q_n(i-1,j) + Q_n(i,j-1), \qquad i,j = 1,\dots,n. \end{aligned} \tag{5}$$

THEOREM 3. *One has*

$$F_1 F_2 \cdots F_n Q_n = P_n^T, \tag{6}$$

and the Cholesky factorization [4] *of Q_n is given by*

$$Q_n = P_n P_n^T. \tag{7}$$

EXAMPLE.

$$Q_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. We define the matrices $Q_n^{(k)}$, $k = 0,\dots,n$, by

$$Q_n^{(k)}(i,j) := \begin{cases} P_n(j,i), & i \leqslant k, \\ Q_n(i,j-k), & i \geqslant k. \end{cases}$$

It is easily verified that $Q_n^{(0)} = Q_n$, $Q_n^{(n)} = P_n^T$, and

$$Q_n(i,j-k) = P_n(j,i) \qquad \text{if} \quad i = k. \tag{8}$$

We show $F_{n-k}Q_n^{(k)} = Q_n^{(k+1)}$: For $i \leqslant k$ the definition of F_{n-k} yields

$$\left(F_{n-k}Q_n^{(k)}\right)(i,j) = P_n(j,i) = Q_n^{(k+1)}(i,j).$$

Let $i > k+1$. By (5) we have

$$\left(F_{n-k}Q_n^{(k)}\right)(i,j) = Q_n(i,j-k) - Q_n(i-1,j-k) = Q_n(i,j-(k+1)). \tag{9}$$

Let $i = k+1$. From (8) and (5), again we obtain (9) as well as

$$\left(F_{n-k}Q_n^{(k)}\right)(k+1,j) = Q_n(k+1,j-(k+1)) = P_n(j,k+1).$$

Thus (6) is proven. The Cholesky factorization (7) now follows from (3). ∎

REMARK. The matrix $F_{n-(k-1)}$ performs the kth Gaussian elimination step for the matrix Q_n.

EXAMPLE.

$$F_3Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{bmatrix}.$$

LEMMA 3.

$$Q_n^{-1} = \left(P_nP_n^T\right) = J_nP_n^TP_nJ_n \tag{10}$$

and

$$Q_n^k = P_nJ_nQ_n^{1-k}J_nP_n^T, \qquad k \in \mathbb{Z}. \tag{11}$$

Proof. By (4), it follows that

$$\left(P_n^T\right)^{-1}P_n^{-1} = (J_nP_nJ_n)^TJ_nP_nJ_n = J_nP_n^TJ_nJ_nP_nJ_n = J_nP_n^TP_nJ_n,$$

whereby (10) is proven. If $k = 1$, (11) reduces to (7). With (10) and (7) in mind, we perform the induction steps:

$$
\begin{aligned}
Q_n Q_n^k &= P_n J_n J_n P_n^T Q_n^k = P_n J_n (J_n P_n^T P_n J_n) Q_n^{1-k} J_n P_n^T \\
&= P_n J_n Q_n^{1-(k+1)} J_n P_n^T
\end{aligned}
$$

and

$$
\begin{aligned}
Q_n^{-1} Q_n^k &= Q_n^{-1} P_n J_n Q_n^{-1} Q_n^{1-(k-1)} J_n P_n^T \\
&= Q_n^{-1} P_n J_n J_n P_n^T P_n J_n Q_n^{1-(k-1)} J_n P_n^T \\
&= P_n J_n Q_n^{1-(k-1)} J_n P_n^T . \qquad \blacksquare
\end{aligned}
$$

By carrying out the multiplication of the matrix equation (7) we get an identity for the binomial coefficients:

COROLLARY 1.

$$
\binom{i+k}{k} = \sum_{l=0}^{n} \binom{i}{l}\binom{k}{l}, \qquad i, k = 0, \dots, n.
$$

Corollary 1 can also be derived from the *Vandermonde convolution formula* [3]

$$
\binom{n}{m} = \sum_{k=0} \binom{n-p}{m-k}\binom{p}{k}.
$$

REMARK. The diagonal entries of the matrix Q_n are essentially the Catalan numbers [3], which are defined as

$$
c_k := \frac{1}{k+1}\binom{2k}{k}.
$$

Therefore we have

$$
Q_n(k,k) = \sum_{l=0}^{k} \binom{k}{l}^2 = \binom{2k}{k} = (k+1)c_k, \qquad k \geqslant 0.
$$

If we look at the elements of the matrix equation $I_n = Q_n(J_n P_n^T P_n J_n)$, we find

COROLLARY 2.

$$\sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^{k+j} \binom{i+k}{k} \binom{l}{k} \binom{l}{j} = \delta_{ij}, \qquad i, j = 0, \dots, n.$$

Corollaries 1 and 2 yield

COROLLARY 3.

$$\sum_{k=0}^{n} \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{k+j} \binom{i}{m} \binom{k}{m} \binom{l}{m} \binom{l}{j} = \delta_{ij}, \qquad i, j = 0, \dots, n.$$

From the definition of P_n we know that $\det P_n = 1$ and, utilizing (7), that also $\det Q_n = 1$. Thus P_n and Q_n are elements of $SL(n+1, \mathbb{Z})$, the group of matrices in $\mathbb{Z}^{(n+1)\times(n+1)}$ with determinant 1. Furthermore, all eigenvalues of Q_n are positive by the Cholesky factorization (7). For the spectrum σ of Q_n, we have

$$\sigma(Q_n) = \sigma(P_n P_n^T) = \sigma(P_n^T P_n) = \sigma(J_n P_n^T P_n J_n) = \sigma(Q_n^{-1}).$$

Thus, if λ is an eigenvalue of Q_n, then $1/\lambda$ is one, too. It follows that 1 is an eigenvalue if the dimension of Q_n is odd. Because the eigenvectors are calculated through a finite number of rational operations and because $Q_n \in \mathbb{Z}^{(n+1)\times(n+1)}$, the elements of the eigenvector corresponding to the eigenvalue 1 can be represented as integers. The eigenvalue equation $Q_n \xi = \xi$, n even, yields

THEOREM 4. *If n is even, the diophantine system of equations*

$$\sum_{k=0}^{n} \binom{i+k}{k} \xi_k = \xi_i, \qquad i = 0, \dots, n,$$

has nontrivial solutions in $\mathbb{Z}$.

The following table shows the nontrivial solutions, the components of which have no common divisors, for $n = 2, 4, 6, 8$, and 10:

n	ξ
2	$(2, 1, -1)^T$
4	$(14, 7, -3, -8, 4)^T$
6	$(6, 3, -1, -3, -1, 3, -1)^T$
8	$(2002, 1001, -299, -949, -467, 581, 721, -784, 196)^T$
10	$(156, 78, -22, -72, -40, 33, 59, -6, -66, 45, -9)^T$

Let us consider again the Pascal matrix P_n. It turns out that there is a short formula for the elements of all powers of P_n. If for convenience we set $0^0 := 1$, then

THEOREM 5. *One has*

$$P_n^k(i,j) = \binom{i}{j} k^{i-j} \qquad \textit{for} \quad i,j = 0,\dots,n \textit{ and } k \in \mathbb{Z}$$

or

$$P_n^k = W_k P_n W_n^{-1} \qquad \textit{for} \quad k \in \mathbb{Z} \setminus \{0\},$$

where $W_k := \operatorname{diag}(1, k, k^2, \dots, k^n)$.

Proof. Since W_k is nonsingular for $k \neq 0$, the second statement follows immediately from the first one. For $k = 0, 1$, and -1 the first statement holds by the definition of P_n and (4). Let $\sigma \in \{1, -1\}$. Thus by (4) and the definition of the matrix product

$$\begin{aligned}
P_n^{k+\sigma}(i,j) &= P_n^{\sigma} P_n^k(i,j) = \sum_{l=j}^{i} \sigma^{i+l} \binom{i}{l}\binom{l}{j} k^{l-j} \\
&= \sum_{r=0}^{i-j} \binom{i}{r+j}\binom{r+j}{j} \sigma^{i+j+r} k^r = \sum_{r=0}^{i-j} \binom{i}{j}\binom{i-j}{r} \sigma^{i-j-r} k^r \\
&= \binom{i}{j}(k+\sigma)^{i-j}.
\end{aligned}$$

Now the statement is obvious by induction. ■

From now on we let e_i be the ith unit vector in $\mathbb{R}^{n+1}$, $i = 0, \ldots, n$, and $e := (1, \ldots, 1)^T \in \mathbb{R}^{n+1}$ the summation vector. It is well known that the sums of the rows of the Pascal matrix P_n are powers of 2. This fact can be generalized for all powers of the Pascal matrix (see also [1]). As a corollary to Theorem 5 we get

LEMMA 4. (Swapping lemma).

$$\left(P_n^k e\right)_i = e_i^T P_n^k e = \sum_{j=0}^{i} \binom{i}{j} k^{i-j} = (k+1)^i \quad \textit{for} \quad k \in \mathbb{Z} \textit{ and } i = 0, \ldots, n.$$

The swapping lemma states that the roles of the base and the exponent are interchangeable; thus the term "swapping."

As a first consequence we have

COROLLARY 4.

$$\sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (l+1)^k = 0 \qquad \textit{if} \quad p > k.$$

Proof. First we state that, for any square matrix A having nonzero entries only beneath the diagonal, the first k rows of A^k are always zero. Thus, if $k < p$, the swapping lemma yields for every $n \geqslant p$

$$0 = e_p^T (P_n - I_n)^k e = \sum_{l=0}^{p} \binom{p}{l} (-1)^{p-l} e_i^T P_n^l e$$

$$= \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (l+1)^k. \qquad \blacksquare$$

We are now able to give an explicit formula for the sum of the kth powers.

THEOREM 6. *For* $k \geqslant 0$, $n \geqslant 1$

$$\sum_{m=1}^{n} m^k = \sum_{p=0}^{k} \binom{n}{p+1} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (l+1)^k.$$

Proof. By the swapping lemma we have

$$\begin{aligned}
\sum_{m=1}^{n} m^k &= \sum_{l=0}^{n-1} e_k^T P_n^l e = \sum_{l=0}^{n-1} e_k^T (P_n - I_n + I_n)^l e \\
&= \sum_{l=0}^{n-1} \sum_{p=0}^{l} \binom{l}{p} e_k^T (P_n - I_n)^p e = \sum_{l=0}^{n-1} \sum_{p=0}^{n} \binom{l}{p} e_k^T (P_n - I_n)^p e \\
&= \sum_{p=0}^{n} \left[\sum_{l=0}^{n-1} \binom{l}{p} \right] e_k^T (P_n - I_n)^p e = \sum_{p=0}^{n} \binom{n}{p+1} e_k^T (P_n - I_n)^p e \\
&= \sum_{p=0}^{n} \binom{n}{p+1} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} e_k^T P_n^l e \\
&= \sum_{p=0}^{n} \binom{n}{p+1} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (l+1)^k \\
&= \sum_{p=0}^{k} \binom{n}{p+1} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (l+1)^k .
\end{aligned}$$

In the last step Corollary 4 enabled us to reduce the summation to k summands, the values of which depend only on n for fixed k. ■

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